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ANNALS OF MATHEMATICS.

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EFFECT OF FRICTION AT CONNECTING-ROD BEARINGS ON THE FORCES TRANSMITTED.

By PROFS. J. BURKITT WEBB* and D. S. JACOBUS, Hoboken, N. J.

Fig. 1 represents the equilibrium of the forces acting on a connecting rod, with a centre line WC and wrist and crank pins indicated by circles about W and C whose radii are respectively r_w and r_c.

Figs. 1′ and 1″ represent the equilibrium of the forces acting at the wrist and crank pins.

The forces shown in these figures are as follows:

$P_a - F_1$ = pressure of steam less the accelerating force for piston, piston rod, and cross-head;

G' = the pressure of the cross-head guides against the cross-head for frictionless pins, acting at the centre of the wrist pin at the angle $90° + \varphi'$. This angle is the same for frictionless or rough pins;†

G'_f = the pressure when there is friction.

P_c and P_w = pressures of the pins C and W against the rod for frictionless pins;

P_{cf} and P_{wf} = the same when there is friction;

$KG = PH$ = the single force which, if applied in the line NG, will support the weight and correctly accelerate the rod;

I and J = pressures of the ends of the rod at the pin centres W and C due to its acceleration and weight and for frictionless pins;

I' and J' = the same pressures for rough pins acting at the points F and E of the pins, F being the foot of a perpendicular from W on P_{wf} and E the foot of a perpendicular from C on P_{cf}.

The moments of I and I' about N must evidently be equal and opposite to those of J and J' about the same point; from which it follows that

$$I : J = CN : NW$$

and

$$I' : J' = EN'' : N''F.$$

*Professor Jacobus insists upon my name appearing first in this article. I fully appreciate the courtesy, but it is hardly fair to himself, as he has done most of the work. J. B. W.

†φ' = angle of friction at cross-head guides.

In addition to the above the following lines will be needed for the purpose of demonstration:

Ec and *Ff* friction circles about *C* and *W*, the radii of which are respectively equal to $r_c \sin \varphi$ and $r_w \sin \varphi$;

FE and *FC* connecting *F* with *E* and *C*;

MD from *M*, parallel and equal to *PK* or *GH*;

DO connecting *D* with *O*.

Fig. 2 is composed of various equilibrium polygons representing the equilibrium of the forces acting on various parts, each polygon being drawn with and without friction; these forces are, of course, equal and parallel to those in Fig. 1, *nm* being the force due to the weight and acceleration and therefore the same as *KG* and *PH*. Fig. 2′ is an enlargement, for the sake of clearness, of a portion of Fig. 2.

smo is the equilibrium triangle of the forces acting on the wrist pin when there is no friction on wrist or crank pin.

mno is the equilibrium triangle of the forces acting on the rod, as a whole, for frictionless pins, and *mnh* the same when there is friction. The introduction of the force *oq* transmitted by the rod from wrist to crank pin divides *mn* into *mq* and *qn*, forces applied at the wrist and crank pins, thereby cutting *mno* into two polygons, *mqo* representing the equilibrium of the forces acting on the crank pin and *oqn* representing forces on the wrist pin. In the same way *hp* divides the polygon *mnh* into *mph* and *hpn*.

mjo and *njo*, and *mhiq* and *qihn*, are also polygons for wrist and crank pins.*

In addition to the above the following lines will be needed for the purpose of demonstration:

ma perpendicular to P_{cf};

af parallel to *CF* in Fig. 1;

hf perpendicular to P_{cf};

nc perpendicular to P_{cf};

ch parallel to *EF* (Fig. 1) through *c*, the intersection of *nc* and *af*, and passing therefore through *h*, as will be shown;

mxz perpendicular to P_{cf} through *m*;

zc parallel to *mn* through *z*, the intersection of *mx* with *ef*;†

* *mq* and *qn* are the simplest set of forces that will support the weight and produce the required acceleration, inasmuch as they cause no unnecessary tension in the rod. Any pair of forces having *mn* for a resultant will give the support and acceleration, consequently we may make various convenient suppositions in regard to these forces, as, for instance, we may suppose that the crank-pin force remains normal to the crank circle, which gives us *mj* and *jn*. The intersection of any such pair of forces will fall on the line *L′S′*, and any pair may be changed to another by introducing two equal and opposite forces at the ends of the rod, thus *qj* combined with *mq* produces *mj* and *jq* combined with *qn* produces *jn*.

† By mistake the figure contains two *c*'s.

nwv perpendicular to P_{wf} through n;

vf' parallel to mn through v, the intersection of nw with cf;

hd perpendicular to P_{wf};

hi perpendicular from h on $L'S'$, which is parallel to the centre line LS, Fig. 1;

bc perpendicular on $L'S'$ from b, the intersection of hd with af;

hg perpendicular from h on cf;

bk perpendicular from b on hi;

$bkgh$ a semicircle containing the right angles bkh and bgh;

kg a line completing the triangle ghk which is similar to dab, similar to WCF (Fig. 1), as will be proved;

nq' through n parallel to the crank radius;

mj connecting m with j the intersection of nq' with ao;

or' perpendicular from o on nq';

hq' perpendicular from h on nq'.

Fig. 2 contains the following forces:

$sm = P_a - F_1$;

om and $hm = P_w$ and P_{wf};

on and $hn = P_c$ and P_{cf};

os and $hs = G'$ and G'_f, the guide reactions;

mn = the resultant of the weight of the rod at its centre of gravity and the forces due to the acceleration of the different parts of the rod. nm is therefore that single force which, if applied at the proper point, would support and correctly accelerate the rod; nm is also the resultant of such parts of the wrist and crank pin pressures as give the support and acceleration;

mp and mq and pn and qn = the parts of the wrist and crank pin pressures which actually give the support and acceleration;

hp and oq = the force transmitted from wrist to crank pin by the rod, with and without friction, hp obliquely from F to E, oq axially;

oh = change in guide reaction due to pin friction;

bc and $hk = A$ and $B = \frac{P_{wf} r_w \sin \varphi}{nR}$ and $\frac{P_{cf} r_c \sin \varphi}{nR}$ and

$hi = bc + hk = A + B$, two forces which will be explained in the proper place;

jo = that portion of the force transmitted from the wrist to the crank pin that performs work upon the fly wheel, for frictionless pins;

qi = the force transmitted axially if P_{wf} and P_{cf} be supposed to act at the centres of the pins, and moments be introduced to counterbalance the effect of altering their points of application;

or' = force acting on crank pin in tangential direction for frictionless pins;

hq' = the same when there is friction.

We will suppose the force $P_a - F_1$ applied at the wrist pin W to be known and proceed to determine the force that reaches the crank pin C.

For frictionless pins we proceed as follows:

Construct the polygon $smqo$ by making sm and mq equal and parallel respectively to $P_a - F_1$ and J, and drawing from q and s lines parallel to CW and G', marking their intersection o, then will os be the guide reaction and oq the force transmitted from W to C; om will be P_w the resultant of the forces G' and $P_a - F_1$. Next lay off qn equal and parallel to J and complete the triangle mno, then will $no = P_c$, the pressure of the crank pin upon the connecting rod. This pressure P_c is pressure available for doing work upon the crank and fly wheel.

If there is friction the construction of the diagrams is more complicated, for the introduction of friction not only changes the direction and magnitude of the forces, but also alters their points of application.

There are three conditions that govern the construction of a diagram including the effect of friction; these are:

(a) The resultant of P_{cf} and P_{wf} must equal in magnitude and direction, and have the same line of action, NG, as the resultant of P_c and P_w;

(b) The projections of P_{wf} and P_w on a line perpendicular to the guide reaction must be equal to each other and to that of $P_a - F_1$;

(c) The presence of friction changes the forces P_c and P_w into P_{cf} and P_{wf}, which no longer pass through the centres of the pins, but are tangent to circles Ee and Ff whose radii are constant and respectively equal to $r_c \sin \varphi$ and $r_w \sin \varphi$.

The last of these conditions is generally given by authorities, such as Rankine and Weisbach, and requires therefore no demonstration.*

That the resultant of P_{cf} and P_{wf} should be the same as that of P_c and P_w arises from the fact that either is the single force PH or KG which would support the weight and produce the required acceleration of the rod, if applied at the proper point, and must therefore be independent of the friction of the pins.

The second of these conditions is evident from Fig. 2, where mos is the triangle of forces acting on a frictionless wrist pin and mhs the triangle with friction considered; obviously P_w and P_{wf} have the same component perpendicular to os as $P_a - F_1$ has.

If we attempt to apply these principles directly to Fig. 1, they lead respectively to the following geometrical conditions:

(a) HP must be equal in length and direction to GK and must lie in the same line GN;

(b) DO is parallel to G';

*See Rankine's Machinery and Mill Work, p. 428.

(*c*) PF is perpendicular to $FW = r_w \sin \varphi$, and PE is perpendicular to $EC = r_c \sin \varphi$.

The construction of the diagram under these conditions can only be accomplished by approximate methods more difficult of application than those which we will explain in the construction of Fig. 2.

The geometrical conditions for the construction of Fig. 2 are as follows:

(*a*) *mn* is common to both the polygons *mnos* and *mnhs*;

(*b*) *ohs* is a right line.

(*c*) In applying the third principle we meet with the only difficulty in the construction of Fig. 2; a fundamental principle in mechanics, however, furnishes us with a simple means of solving the problem with all the accuracy desirable.

Inasmuch as the effect of friction is to displace the forces which the pins exert upon the rod, so that they become tangent to the friction circles previously mentioned, these forces will exert moments upon the rod tending to rotate it, and therefore affecting the other forces of the system, so that the forces P_c and P_w become P_{cf} and P_{wf}, besides becoming displaced from C to E and W to F. By a principle in mechanics a force applied at any point, as F, is equivalent to an equal force, in magnitude and direction, at any other point, as W, plus a moment equal to the force multiplied by the perpendicular distance through which it has been displaced. We may, therefore, suppose the forces P_{cf} and P_{wf} to be applied at the centres of the pins if, at the same time, we introduce the moments

$$P_{wf} r_w \sin \varphi \quad \text{and} \quad P_{cf} r_c \sin \varphi.$$

For convenience we will suppose each moment to be produced by a pair of equal and opposite forces acting perpendicular to the rod at W and C. Calling these forces A and B, and letting the distance $WC = nR = n \times$ crank radius, we shall have

$$AnR = P_{wf} r_w \sin \varphi,$$

and

$$BnR = P_{cf} r_c \sin \varphi,$$

so that the moment to be introduced will be

$$AnR + BnR = (A+B)nR.$$

According to this, instead of supposing P_{wf} and P_{cf} to act on the rod at F and E, no error will be involved if we suppose them to act at W and C and introduce two additional forces, $A + B$ acting at W perpendicular to WC and $-(A + B)$ acting at C. A and B appear also in the equilibrium polygons Figs. 3 and 4.*

*When the angle β is increasing A is $+$. B is $+$ or $-$ according to whether θ is supposed to increase or diminish.

The important advantage of this change is that it allows the use of the known accelerating forces I and J in place of the unknowns I' and J'.

The construction of Fig. 2 by the aid of $A + B$ is as follows:

Having laid off $P_a - F_1$ and mq as before, and drawn through q and s lines qo and so, respectively parallel to the rod and the guide reaction, cut so by a line parallel to qo and at a distance from it $= A + B$. This will give the point h from which the perpendicular $hi = A + B$ may be let fall upon qo. The polygon $smqih$ will now represent the equilibrium of the forces acting on the wrist pin, and hs will be the new value of the guide reaction. Connecting h with m and n we obtain P_{wf} and P_{cf}.

The value of $A + B$ can be obtained with an exactness amply sufficient by using the values of P_w and P_c in place of those of P_{wf} and P_{cf}, as will be shown in the case of a horizontal high-speed engine, to which formulæ derived by this method have been applied. This form of the result has the advantage that it enables the same to be easily applied.

As the above reasoning may seem to involve a departure from the exact conditions of the problem, in which the forces are really applied at F and E, we will proceed to prove that the introduction of A and B into the solution for frictionless pins will give the correct result when friction is considered, that is;

To prove that $hi = A + B$.

This proof divides itself into three parts as follows:

(a) to prove that $cb = A$;

(b) to prove that $kh = B$;

(c) to prove that $ih = kh + cb$.

(a) From the similar triangles CFW and abd we have

$$\frac{WF}{CW} = \frac{db}{ad};$$

but

$$ad = \frac{P_{wf}}{\cos \mu},$$

because am and dh are perpendicular to mh, and ad makes with mh the angle μ, also

$$CW = nR,$$

and

$$WF = r_w \sin \varphi,$$

substituting these values, we have

$$db = \frac{P_{wf} r_w \sin \varphi}{nR \cos \mu};$$

but

$$cb = db \cos \mu;$$

$$\therefore cb = \frac{P_{wf} r_w \sin \varphi}{nR} = A.$$

(*b*) To prove that $kh = B$.

To prove this we must first show the following:

I. That a line through e parallel to EF divides mn at p into the two forces $mp = I'$, and $pn = J'$ acting at F and E.

II. That the line ep produced will pass through h.

III. That hgk is similar to CWF.

I. To show that mn is correctly divided at p, that is to show that

$$mp = I', \text{ and } pn = J'.$$

The relations previously given between I, I', J, and J' are

$$I + J = I' + J' = mn,$$

$$I : J = CN : NW,$$

and

$$I' : J' = EN'' : N''F.$$

By construction, therefore, $mn = I' + J'$, and we have only to prove that

$$mp : pn = I' : J'.$$

First, to prove that

$$mr : nr = CN : NF';$$

q being the point of division for frictionless pins, we know that

$$mq : qn = CN : NW,$$

and also, from the similar triangles amq and αnq, we have

$$mq : qn = aq : q\alpha;$$

consequently

$$aq : q\alpha = CN : NW.$$

This, in addition to the similarity of the triangles CFW with $a\pi\alpha$, CFF' with $a\pi f'$, and $CN'N$ with arq, gives us two similar figures contained in Figs. 1 and 2; viz:

$$CN'FF'WNC,$$

similar to

$$ar\pi f'\alpha qa,$$

from which it follows that

$$CN : NF' = aq : qf'.$$

But

$$aq : qf' = ar : r\pi,$$

because arq and $a\pi f'$ are similar, and by the similar triangles mra and $nr\pi$ we have

$$ar : r\pi = mr : nr;$$

$$\therefore CN : NF' = mr : nr,$$

which was to be proved.

Second, to prove that

$$mp : pn = EN'' : N''F.$$

From the similar triangles *ncr* and *mzr* we have

$$mr : rn = zr : rc;$$

but

$$mr : rn = CN : NF'$$
$$= CN' : N'F;$$
$$\therefore zr : rc = CN' : N'F.$$

This, in addition to the similarity of the triangles $FE'C$ with ecz, FEC with exz, and CEE' with zxc, gives us two similar figures contained in Figs. 1 and 2; viz:

$$FN'CE'EN''F$$

similar to

$$crzcxpe,$$

from which it follows that

$$xp : pc = EN'' : N''F.$$

From the similar triangles *mxp* and *ncp* we have

$$mp : pn = xp : pc;$$
$$\therefore mp : pn = EN'' : N''F;$$

but

$$EN'' : N''F = I' : I';$$
$$\therefore mp : pn = I' : I',$$

which was to be proved.

II. To show that *cp* passes through *h*.

At F the force P_{ef} of the pin against the rod is divided into I' and the force transmitted from F to E, or, in other words, P_{ef} is the resultant of the latter forces, and when reversed will form with them a triangle of forces. This triangle *mhp* appears in Fig. 2, in which $pm = I'$ and $mh = -P_{ef}$, and the third side, having been drawn through p and parallel to FE, must pass through h, so as to make hp the force transmitted from F to E.

III. To show that *hgk* is similar to CWF.

The angle kgh is equal to $hgb + bgk = 90° + \mu$, because hgb is inscribed in the semicircle bgh and $bgk = bhk = \mu$; also, khg is equal to $fao = \psi$, because gh and kh are perpendicular to fa and oa; therefore the triangle hkg has two of its angles the same as the angles of CWF, in which $FCW = \psi$ and $CWF = 90° + \mu$, it being the exterior angle of the right angle triangle WFS.

We are now prepared to prove that $kh = B$. From the similar triangles FEC and chf we have

$$\frac{hf}{cf} = \frac{CE}{CF};$$

but $$CE = r_c \sin \varphi,$$

and $$cf = \frac{P_{cf}}{\cos a};$$

$$\therefore hf = \frac{P_{cf} r_c \sin \varphi}{CF \cos a};$$

but $$gh = hf \cos a;$$

$$\therefore gh = \frac{P_{cf} r_c \sin \varphi}{CF}.$$

From the similar triangles hgk and CWF we have

$$\frac{gh}{kh} = \frac{CW}{CF} = \frac{nR}{CF};$$

$$\therefore kh = gh \frac{CF}{nR}.$$

Substituting the value of gh and reducing, we have

$$kh = \frac{P_{cf} r_c \sin \varphi}{nR} = B,$$

as was to be proved.

(*c*) To prove that $ih = kh + cb$.

This follows because bk is perpendicular to hi, and therefore $bc = ki$; consequently, as

$$cb = A \quad \text{and} \quad kh = B,$$

$$ih = A + B,$$

as was to be proved.

ANALYTICAL EXPRESSION OF THE FOREGOING PRINCIPLES.

Let R = radius of crank circle;

nR = length of connecting rod;

lR = distance from the wrist pin to the foot of the perpendicular let fall from the centre of gravity of the rod upon its centre line;

cR = distance of the centre of gravity from the centre line of the rod;

θ and β = angles of crank and connecting rod with the path of the wrist pin;

δ = angle made by tipping the engine up about the crank shaft;

$\tan \varphi$ = coefficient of friction at wrist and crank pins;

$\tan \varphi'$ = coefficient of friction at cross-head guides;

ω = angular velocity of crank;

W_1 and M = weight and mass of the piston, piston rod, and cross-head;

W_2 and m = weight and mass of the connecting rod;

C = component of the weight of the piston, piston rod, and cross-head that acts in the direction of the centre line of the cylinder;

D and E = portions of the weight of the connecting rod borne respectively by the wrist and crank pins;

H = friction of the piston and piston rod;

X_1 and X_2 = components, in the direction of the line of travel of the wrist pin, of the accelerating forces at the wrist and crank pins;

Y_1 and Y_2 = components of the accelerating forces at the wrist and crank pins at right angles to X_1 and X_2;

P_n = force produced by the pressure of the steam on the piston;

F_1 = force required to produce the acceleration of the mass of the piston, piston rod, and cross-head;

G and G_f = normal component of the reaction of the cross-head guides for frictionless and rough pins;

N_f and T_f = components of P_{nf} parallel and perpendicular to the crank;

T_f' = the latter reduced to the centre of the crank pin.

We now have $$C = W_1 \sin \delta.$$

The parts of the weight of the connecting rod supported at wrist and crank pins are

$$D = \frac{n - l + c \tan (\beta - \delta)}{n} W_2,$$

and

$$E = \frac{l - c \tan (\beta - \delta)}{n} W_2.$$

Combining these forces with P_n, the accelerating forces, and those due to friction, we obtain the equilibrium polygons, represented in Figs. 3 and 4, for the forces acting on the wrist and crank pins.

From these polygons we obtain the following values of the several forces:*

$$G_f = \frac{1}{1 + \tan \varphi' \tan \beta} \left\{ \begin{pmatrix} P_n + C - F_1 - H + D \sin \delta \\ -(A + B) \sin \beta - X_1 \end{pmatrix} \tan \beta + Y_1 + D \cos \delta - (A + B) \cos \beta \right\}$$

$$P_{nf} = \sqrt{[(P_n + C - F_1 - H - G_f \tan \varphi')^2 + G_f^2]},$$

* The method of obtaining these forces, together with those required to produce the acceleration, will be given in detail in an article yet to be published.

$$T_f = \left.\begin{matrix} P_a + C - F_1 - H - G_f \tan \varphi' \\ + D \sin \delta - (A + B) \sin \beta - X_1 \end{matrix}\right\} \sec \beta \sin (\theta + \beta)$$
$$- Y_2 \cos \theta - X_2 \sin \theta - (A + B) \cos (\theta + \beta) - E \cos (\theta + \delta),$$

$$N_f = \left.\begin{matrix} P_a + C - F_1 - H - G_f \tan \varphi' \\ + D \sin \delta - (A + B) \sin \beta - X_1 \end{matrix}\right\} \sec \beta \cos (\theta + \beta)$$
$$+ Y_2 \sin \theta - X_2 \cos \theta + (A + B) \sin (\theta + \beta) + E \sin (\theta + \delta),$$

$$P_{cr} = \sqrt{(T_f^2 + N_f^2)},$$

$$T'_f = T_f - Bn.$$

APPLICATION OF THE FOREGOING FORMULÆ.

We will apply these formulæ to the case of a horizontal high-speed engine, assuming an excessive coefficient of friction at the wrist and crank pins, and show that the values of A and B are determined by one approximation to as great a degree of accuracy as is desirable.

The dimensions of the engine are

Length of stroke, 12″;

Diameter of cylinder, 10″;

Length of connecting rod, 36″;

Distance from the wrist pin to the centre of gravity of the rod, 20″.15;

Principal radius of gyration, 15″.00;

Diameter of crank pin, 3″;

Diameter of wrist pin, $2\frac{1}{2}$″;

Weight of piston, piston rod, and cross-head, 90 lbs.;

Weight of connecting rod, 70 lbs.;

Sin φ, .24.

Neglecting friction the forces contained in Table I are obtained.

TABLE I.

Forces in pounds per square inch of piston area.

θ	P_a	F_1	Y_1	Y_2	X_1	X_2	G	T	N	P_{cr}	P_c
0	+ 83	+ 20.50	0	0	+ 6.85	+ 7.82	0	0	+ 47.8	+ 62.5	+ 47.8
30	+ 81	+ 16.69	— .50	— 3.33	+ 5.63	+ 6.71	+ 4.43	+ 33.1	+ 41.0	+ 64.4	+ 52.6
50	+ 76	+ 10.81	— .76	— 5.10	+ 3.72	+ 4.89	+ 7.17	+ 51.7	+ 26.4	+ 65.6	+ 58.1
70	+ 54	+ 3.74	— .93	— 6.26	+ 1.41	+ 2.49	+ 6.84	+ 48.4	+ 2.7	+ 50.7	+ 48.4
90	+ 33	— 2.97	— .99	— 6.66	— .85	— .17	+ 5.20	+ 37.0	— 12.9	+ 36.4	+ 39.2
110	+ 20	— 8.28	— .93	— 6.26	— 2.70	— 2.74	+ 4.00	+ 27.9	— 22.1	+ 28.6	+ 35.5
130	+ 1	— 11.79	— .76	— 5.10	— 4.00	— 4.95	+ 1.41	+ 12.0	— 19.5	+ 12.9	+ 22.9
150	— 25	— 13.74	— .50	— 3.33	— 4.78	— 6.54	— 1.04	— 2.4	— 1.4	— 11.3	— 2.7
180	— 80	— 14.64	0	0	— 5.17	— 7.48	0	0	+ 52.7	— 65.4	— 52.7

As a first approximation; i. e. letting

$$A_1 = \frac{P_w r_w \sin \varphi}{nR} \quad \text{and} \quad B_1 = \frac{P_c r_c \sin \varphi}{nR},$$

we obtain the values of A and B given in Table II.

TABLE II.

Forces in pounds per square inch of piston area.

θ	A_1	B_1	$A_1 + B_1$	θ	A_1	B_1	$A_1 + B_1$
0	+ .521	+ .478	+ .999	110	— .238	+ .355	+ .117
30	+ .537	+ .526	+ 1.063	130	— .108	+ .229	+ .121
50	+ .547	+ .581	+ 1.128	150	— .094	+ .027	— .067
70	+ .422	+ .484	+ .906	180	— .545	+ .527	— .018
90	0	+ .392	+ .392				

Making use of the values of A and B given in Table II, we obtain the forces when friction is included as given in Table III.

TABLE III.*

Forces in pounds per square inch of piston area.

θ	G_f	$G - G_f$	N_f	$N - N_f$	T_f'	$T - T_f'$	P_{cf}	$P_c - P_{cf}$	P_{wf}	$P_w - P_{wf}$
0	— 1.0	+ 1.00	+ 47.8	0	— 3.9	+ 3.88	+ 47.8	— .01	+ 62.5	+ .01
30	+ 3.4	+ 1.06	+ 41.5	— .53	+ 29.0	+ 4.08	+ 52.5	+ .13	+ 64.4	+ .06
50	+ 6.0	+ 1.13	+ 27.3	— .87	+ 47.6	+ 4.22	+ 57.9	+ .18	+ 65.5	+ .11
70	+ 5.9	+ .92	+ 3.6	— .86	+ 45.2	+ 3.21	+ 48.2	+ .24	+ 50.6	+ .11
90	+ 4.8	+ .40	— 12.5	— .40	+ 34.6	+ 2.35	+ 39.1	+ .13	+ 36.4	+ .05
110	+ 3.8	+ .12	— 22.0	— .11	+ 25.8	+ 2.06	+ 35.5	+ .07	+ 28.6	+ .05
130	+ 1.3	+ .11	— 19.4	— .09	+ 10.7	+ 1.28	+ 22.9	+ .04	+ 12.9	+ .01
150	— .9	— .09	— 1.4	+ .03	— 2.5	+ .21	— 2.7	.00	— 11.3	.00
180	.0	+ .02	+ 52.7	0	— 3.0	+ 3.18	— 52.7	.00	— 65.4	.00

*The computation of this table is facilitated by employing the following equations:

$$G - G_f = (A + B) \sec \beta,$$

$$N - N_f = (A + B) \tan \beta \cos (\theta + \beta) - (A + B) \sin (\theta + \beta),$$

$$T - T_f' = (A + B) \tan \beta \sin (\theta + \beta) + (A + B) \cos (\theta + \beta) + Bn,$$

$$P_w - P_{wf} = \frac{P_w^2 - P_{wf}^2}{P_w + P_{wf}},$$

$$P_c - P_{cf} = \frac{P_c^2 - P_{cf}^2}{P_c + P_{cf}}.$$

Let the values of A and B as determined by a second approximation be A_2 and B_2; then will

$$A_1 - A_2 = \frac{(P_w - P_{wf})r_w \sin \varphi}{nR},$$

and

$$B_1 - B_2 = \frac{(P_c - P_{cf})r_c \sin \varphi}{nR}.$$

Table IV contains values of $A_1 - A_2$ and $B_1 - B_2$.

TABLE IV.

Forces in pounds per square inch of piston area.

θ	$A_1 - A_2$	$B_1 - B_2$	θ	$A_1 - A_2$	$B_1 - B_2$	θ	$A_1 - A_2$	$B_1 - B_2$
0	.0001	.0001	70	.0009	.0024	130	.0001	.0004
30	.0005	.0013	90	.0004	.0013	150	.0000	.0000
50	.0009	.0018	110	.0004	.0007	180	.0000	.0000

The greatest difference between the values of A and B, as obtained by a first and second approximation, is for $\theta = 70°$, at which position $(A_1 + B_1) - (A_2 + B_2) = .0033$ lbs. per square inch of piston area, or $.0036(A + B)$.

The effect of such an error in $A + B$ upon the result may be found as follows:

The introduction of the forces $(A + B) = .905$ for $\theta = 70°$ alters the value of P_c by the amount of .24 lbs. per square inch of piston area; a variation of $A + B$ equal to $.0036(A + B)$ will, therefore, cause this difference in P_c to vary about .0036 times itself, or .0009 lbs. per square inch of piston area. This is equivalent to a variation in P_{cf} of .000019 P_{cf}; i. e. *the value of P_{cf} as obtained by one and two approximations differs by about .00002 times itself.* The variation in P_{wf} for $\theta = 70°$, obtained by a similar method of reasoning, is about .000007 times itself. It is evident, therefore, that the value of $A + B$ as determined by the first approximation is sufficiently accurate.

A BRIEF CONTROL FOR GENERAL SOLUTIONS OF NORMAL EQUATIONS.*

By Mr. A. S. Flint, Washington, D. C.

If we represent a set of normal equations in the usual manner, we have

$$\begin{aligned} 0 &= [an] + [aa]x + [ab]y + [ac]z + \ldots \\ 0 &= [bn] + [ab]x + [bb]y + [bc]z + \ldots \\ 0 &= [cn] + [ac]x + [bc]y + [cc]z + \ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots; \end{aligned}$$

where the equations are continued in similar form until their number m is the same as the number of unknown quantities.

Ordinarily the absolute terms $[an]$, $[bn]$, $[cn]$, . . . have numerical values which are carried through the solution to obtain the numerical values of the unknowns. But if the weights of the values of the unknowns resulting from the solution are desired, it is convenient to retain the symbolic absolute terms through the elimination, and thus obtain the value of each unknown as the sum of a series of terms in $[an]$, $[bn]$, $[cn]$, ... This is called the *general* solution of the equations. Again, if there is reason to suspect that the absolute terms of the observation-equations, from which the normal equations have been formed, will need correction at some future time, it will be convenient to have the general solution. Or it may happen that, before the numerical values of the absolute terms of the normal equations can be given, it is desired to perform the solution as far as the coefficients are concerned. In this last case the general solution is necessary, and it becomes especially desirable to have some brief test of the correctness of the results, while such a test would not be unacceptable whenever for any reason the general solution is made.

The accuracy of the elimination in the general solution might indeed be tested in the same manner as with numerical values of the unknowns, by substituting back in the normal equations. But since the expression for each unknown will have as many terms as there are unknowns, it is evident that, in general, the number of products to be written down will be m^3; and counting only once products that are repeated, it will be found that, in general, the number of products of coefficients to be computed will be $m^3 - \frac{1}{2}m(m-1)$. A briefer control will now be shown.

* Read before the Mathematical Section of the Philosophical Society of Washington.

For brevity, let the normal equations be written as follows:

$$\begin{aligned}
0 &= N_1 + A_1x + A_2y + A_3z + \ldots \\
0 &= N_2 + A_2x + B_1y + B_2z + \ldots \\
0 &= N_3 + A_3x + B_2y + C_1z + \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad \underline{\quad} \quad \underline{\quad} \quad \underline{\quad} \\
&\quad S_1 \quad S_2 \quad S_3 \ldots,
\end{aligned}$$

and let S_1, S_2, S_3, . . . represent respectively the sums of the coefficients in the columns under which they stand. The general solution of the normal equations will give values of $x, y, z, \ldots$ which viewed together will be in the normal form, and which may be written as follows:

$$\begin{aligned}
x &= K_1.N_1 + K_2.N_2 + K_3.N_3 + \ldots \\
y &= K_2.N_1 + L_1.N_2 + L_2.N_3 + \ldots \\
z &= K_3.N_1 + L_2.N_2 + M_1.N_3 + \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad \underline{\quad} \quad \underline{\quad} \quad \underline{\quad} \\
&\quad S'_1 \quad S'_2 \quad S'_3 \ldots,
\end{aligned}$$

and let S'_1, S'_2, S'_3, . . . represent respectively the sums of the coefficients in the columns under which they stand.

If these values of x, y, z be substituted in the first of the normal equations, we have

$$\begin{aligned}
0 = N_1 &+ A_1K_1.N_1 + A_1K_2.N_2 + A_1K_3.N_3 + \ldots \\
&+ A_2K_2.N_1 + A_2L_1.N_2 + A_2L_2.N_3 + \ldots \\
&+ A_3K_3.N_1 + A_3L_2.N_2 + A_3M_1.N_3 + \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots;
\end{aligned}$$

where, after N_1, the first line of the right hand member of the equation comes from the value of x, the second line from the value of y, and so on.

This equation must be generally true; that is, with a given set of coefficients in the normal equations, it must be true whatever the numerical values of N_1, N_2, N_3, . . . may be. From the above equation, therefore, we deduce the following equations, m in number, between the coefficients:

$$\begin{aligned}
-1 &= A_1K_1 + A_2K_2 + A_3K_3 + \ldots \\
0 &= A_1K_2 + A_2L_1 + A_3L_2 + \ldots \\
0 &= A_1K_3 + A_2L_2 + A_3M_1 + \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Summing these into one equation we have

$$0 = 1 + A_1(K_1 + K_2 + K_3 + \dots) + A_2(K_2 + L_1 + L_2 + \dots) + A_3(K_3 + L_2 + M_1 + \dots) + \dots,$$

or

$$0 = 1 + A_1S'_1 + A_2S'_2 + A_3S'_3 + \dots.$$

Again, if we substitute the values of $x, y, z, \dots$ in the second normal equation, we shall have evidently the same result as from the first equation, except that $A_1, A_2, A_3, \dots$ will be replaced by $A_2, B_1, B_2, \dots$; and similarly with all the remaining normal equations. Therefore we have a set of m equations, as follows:

$$0 = 1 + A_1S'_1 + A_2S'_2 + A_3S'_3 + \dots$$
$$0 = 1 + A_2S'_1 + B_1S'_2 + B_2S'_3 + \dots$$
$$0 = 1 + A_3S'_1 + B_2S'_2 + C_1S'_3 + \dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$

The coefficients of $S'_1, S'_2, S'_3, \dots$ in these equations are identical with the coefficients of $x, y, z, \dots$ in the normal equations, and are arranged in the same manner. Therefore, summing all the above m equations into one final control-equation, we have

$$0 = m + S_1S'_1 + S_2S'_2 + S_3S'_3 + \dots.$$

The *control* consists, therefore, simply in summing up the columns of coefficients as they stand in the normal equations, and also in the general expressions for the unknown quantities; that is, in their expressions as functions of the absolute terms of the normal equations; multiplying these sums together in corresponding pairs; and adding algebraically the sum of the products to the number of the unknowns to produce the result 0.

If the s-control has been employed in forming the normal equations we have

$$[as] = [aa] + [ab] + [ac] + \dots$$
$$[bs] = [ab] + [bb] + [bc] + \dots$$
$$\dots\dots\dots\dots\dots\dots,$$

which will be true of the numerical values in any particular example according to the exactness with which the computation has been made. The right hand members of the above equations will be the exact values of $S_1, S_2, \dots$, while the $[as], [bs], \dots$ will, in general, be close approximate values of these sums.

For a numerical example we may take the following:

NORMAL EQUATIONS.

$$
\begin{array}{llll}
0 = -\ 88 & +\ 27x & +\ 6y & \\
0 = -\ 70 & +\ 6x & +\ 15y & +\ z \\
0 = -\ 107 & & +\ y & +\ 54z \\
\hline
 & S_1 = +\ 33 & S_2 = +\ 22 & S_3 = +\ 55
\end{array}
$$

We have here nothing to do with the numerical terms -88, -70, and -107, but only with their symbols N_1, N_2, N_3.

GENERAL SOLUTION.

$$
\begin{array}{llll}
x = -\ 0.0407 N_1 & +\ 0.0163 N_2 & -\ 0.0003 N_3 \\
y = +\ .0163 & -\ .0733 & +\ .0014 \\
z = -\ .0003 & +\ .0014 & -\ .0186 \\
\hline
S'_1 = -\ 0.0247 & S'_2 = -\ 0.0556 & S'_3 = -\ 0.0175
\end{array}
$$

CONTROL.

$$
\begin{array}{ll}
m & +\ 3. \\
S_1 S'_1 & -\ 0.815 \\
S_2 S'_2 & -\ 1.223 \\
S_3 S'_3 & -\ 0.962 \\
\hline
 & +\ 3.000 \\
 & -\ 3.000 \\
\hline
\text{Sum} & 0.000
\end{array}
$$

ELEMENTARY NOTES.

By Prof. J. E. Oliver, Ithaca, N. Y.

Under the above title I propose to give, without much regard to their order, short notes upon some of the fundamental processes and concepts that come into actual work or into teaching, including certain questions as to their best forms and expression or mode of presentation, their natural extension, etc. In such of the notes as are developments of ideas possibly first suggested either in Oliver, Wait, and Jones's Algebra, or at the Cornell University Mathematical Seminary, the references O. W. J. and C. M. S. may be used.

I.

GENERAL AND LOGICO-MATHEMATICAL NOTATION.

I have found all of the following principles and general devices to be very helpful in actual work, and some of them, also, in teaching.

1. *Accents and suffixes.*—It is often well to distinguish between two uses of these, viz:

One use is to mark such values or functions as are co-ordinate with those denoted by the unmodified symbols, or such as are derived therefrom without loss of generality; while the other use is to mark particular values or cases, or to impart additional information. The distinction sometimes cannot be observed, but often it promotes neatness and perspicuity, enabling each symbol to tell naturally its own tale.

Now, two considerations point to the assignment of accents (upper or lower, acute or grave) or of points, etc., to denote derived or co-ordinate values, and of subscript numbers for particular values. Thus, for instance, if x, y be variables, and if x', y' be $ax + by$, $mx + ny$, then the particular values $x_1, y_1, x'_1, \ldots$ (which may be regarded as constants) are connected by the relations $x'_1 = ax_1 + by_1, \ldots$.

(a) Usage already shows a tendency in this direction. Thus exponents, root-indices, and indices of differentiation are written above their stems and simply modify without specializing the meaning of the stem-symbols; and again, it is common to express differentiation by points or accents, as $\dot{x}$ for $\frac{dx}{dt}$, or $f''(x)$ for $\frac{\partial^2}{\partial x^2} f(x)$; and so, we may write $f''_{yy}(x, y)$ for $\frac{\partial^3}{\partial x \partial y^2} f(x, y)$, $f''_{yy}(y, x)$ for $\frac{\partial^3}{\partial y \partial x^2} f(y, x)$, etc,

(β) Apparently, the proposed assignment oftenest avoids the undesirable combination of exponents with upper accents on the same stem.

The distinction here made may often help us in devising appropriate symbols, whether for permanent use or only for the moment. Thus, to exhibit when desired the positive or negative character of an expression, we should use suffixes (perhaps $_+$ or $_-$) because they are to give additional information without modifying the previous meaning and laws of the expression; while to denote that the expression had been *made* positive or negative, by reversing its sign if necessary (O. W. J., p. 5), a superior index (say $^+$ or $^-$) would be used.

In the series $x, x', x'', \ldots$, or $x, x^{\backprime}, \ldots$, or $x, x_1, \ldots$, or etc., the initial term x commonly has a certain unique character as compared with the other terms; and so, I think, should x_0 have, as compared with $x_1, x_2, \ldots$, or else x_0 should not be used; and so with zero values of the integers r, s in the series $x_{rs}, x_{r|s}$, etc.

2. *Indices to copulas.*—(a) It is often convenient to modify the copula $=$ so as either (1) to make some other or less definite assertion than that of strict equality, or else (2) to give additional information. These two general cases are analogous to the correspondingly numbered cases in Art. 1. For instance, in the symbol x'_3, the accent only varies the information given, perhaps by referring to different axes from those of x and y; but the suffix 3 makes that information more definite, though perhaps less general, by showing that the quantity indicated not only is an x', but also is that particular value of x', or is the x' of the particular point P_3.

Let us therefore extend to copulas, (as $=$, $>$, etc.), the principle of Art. 1; i. e. let us modify the copula at need by indices, points, etc., in the general case (1), and by subscript numbers or letters in the general case (2).

For instance, using $\doteq$ in the sense "Approaches in value to, as a limit," $U \doteq_{\Delta x} C$ would so naturally mean "U approaches C as a limit, when Δx approaches its limit (viz. 0)," that these words of explanation would hardly be needed.

So $=_3$ might mean "Is equal to, by reason of equation (or article) 3, above;" and the context would sufficiently show if this were the meaning.

As another illustration of how nearly these symbols may tell their own story when one has in mind the general subject-matter involved, we may write, as follows, a familiar cross-ratio proof of Pascal's hexagon-theorem:

Let c be any conic; 123456 the vertices of any inscribed hexagon; and 7, 8, 9 the respective intersections 23.56, 61.34, 45.12; then are 7, 8, 9 collinear.

For $(8\text{–}2673) =_{23} (6\text{–}2153) =_c (4\text{–}2153) =_{12} (8\text{–}2194) =_8 (8\text{–}2693)$;

$\therefore$ 7, 8, 9 are collinear. Q. E. D.

Here, as is customary, (8–2673) denotes the cross-ratio of the pencil 82, 86, 87, 83; and it almost goes without saying that $=_{23}$, $=_c$, and $=_8$ mean, respectively, "Is equal to, by reason of the transversal 23," and "..... of the conic c," and "..... of the pencil whose vertex is 8."

(b) The copula $\doteq$ admits of further specialization. Between its meaning as first used by me, "Is nearly equal to," and Byerly's modification "Approaches as a limit," and perhaps yet other useful shades of interpretation, the context will sufficiently determine; but another distinction affects its formal laws and therefore should be indicated. The statement $U \doteq V$ may mean (α) that $U - V$ approaches zero, or contains only terms beyond a certain order, or may be for the moment neglected; or it may mean (β) that $(U-V)/\sqrt{UV}$, $= \sqrt{U/V} - \sqrt{V/U}$, does so. Understood in the sense $\left\{\begin{matrix}\alpha\\ \beta\end{matrix}\right\}$, the statement $U \doteq V$ means that the $\left\{\begin{matrix}\text{absolute}\\ \text{relative}\end{matrix}\right\}$ error of taking V for U or U for V may be neglected; and hence the statement is equivalent to $\left\{\begin{matrix}U-V \doteq 0 \doteq V-U\\ U/V \doteq 1 \doteq V/U\end{matrix}\right\}$. The two senses are identical, except that when U and V approach $\left\{\begin{matrix}0\\ \infty\end{matrix}\right\}$ we may have $U \doteq V$ in the sense $\left\{\begin{matrix}\alpha \text{ and not } \beta\\ \beta \text{ and not } \alpha\end{matrix}\right\}$. But, since the ratios of magnitudes are important oftener than the absolute values, let us keep $\doteq$ in the sense (β) and write $\overset{A}{=}$, when necessary, in the sense α, as suggestive of *absolute* error.

Again, to mark the degree of approximation we may write $\overset{1}{=}$, $\overset{2}{=}$, ... and $\overset{A1}{=}$, $\overset{A2}{=}$, ... Thus, $U \overset{An}{=}_h V$ or $U \overset{n}{=}_h V$ would mean that $U - V$ or $(U-V)/\sqrt{UV}$ was of at least the nth order of smallness or negligeability, h being of the first order, or that it had the factor h^n. The suffix could be omitted, and the index n might be fractional or negative: e. g. $\overset{\frac{1}{2}}{=}$. When n is positive, the copula is replaceable by the less explicit $\doteq$ or $\overset{A}{=}$; and any of these copulas may replace their common limiting form $=$.

When n is negative, $U \overset{An}{=} V$ means of course that $U - V$ is not an infinite of more than the nth order. But the definition of $\overset{n}{=}$ connects itself naturally with that of "Relative Error." To define the latter in the usual way, as the ratio of the absolute error of an assumed value to the true value, seems to me open to three objections, viz: It ignores the actual symmetry of relation between true and assumed values, assigning different relative errors to an assumed value and its reciprocal or to estimates k-fold too large and k-fold too small; it makes needlessly inaccurate the rule that the relative error of a product is the sum of the relative errors of the factors; and when the relative errors are far from zero, it may exhibit untruly even their general order of importance, assigning nearly

the same relative error, — 1, whether an estimate be .001 or .000001 of the true value. And these objections are of practical importance when, as frequently happens, we would deal with actual magnitudes the probable errors of whose logarithms may be several units.

To meet the first two objections, we must divide the absolute error not by the true value, but by that symmetric function of the true and the assumed value which gives for quotient the absolute error of the (say, Naperian) logarithm of the assumed value. When both values are real, this divisor lies always between their arithmetical and geometric mean and nearer to the geometric, being one third of the way between them when the relative error vanishes, and becoming $\frac{2}{\pi}$ of their arithmetico-geometric mean when the relative error becomes infinite. The divisor hardly differs by over four per cent. from the geometric mean, unless the relative error defined by it exceed 1.

But the third objection is not thus quite removed; for when the true and assumed values are very disproportionate, their ratio tends toward an infinite of infinitely higher order than the relative error as thus defined. Probably we still can best define "Relative Error" as the absolute error of the Naperian logarithm, while defining $\overset{n}{=}$ not quite analogously, but as above, so that $U \overset{n}{=}_h V$ may mean that $1\sqrt{U/V} - 1\sqrt{V/U}$ is no infinite of more that the order n, whence the ratio U/V is no infinite or infinitesimal of higher order than the infinitesimal h^{2n}: indeed, the two definitions agree as long as n is positive.

If, for a moment, we speak of these approximate equalities as "R-equalities" and "A-equalities," we can say that:—

(1°) When the two members of an $\left\{\begin{matrix} R \\ A \end{matrix}\right\}$-equality are $\left\{\begin{matrix} \text{multiplied or} \\ \text{increased or} \\ \text{diminished} \\ \text{divided} \end{matrix}\right\}$ by any same number, or by the corresponding members of another $\left\{\begin{matrix} R \\ A \end{matrix}\right\}$-equality with the same or higher index, the suffix, whether written or understood, remaining the same; or when they are $\left\{\begin{matrix} \text{raised to any finite} \\ \text{power} \\ \text{multiplied by any} \\ \text{finite number} \end{matrix}\right\}$ whether positive, negative, or imaginary, the index of the resulting $\left\{\begin{matrix} R \\ A \end{matrix}\right\}$ equality is not diminished; neither is it, if corresponding members be $\left\{\begin{matrix} \text{added, without diminishing} \\ \text{multiplied, without increasing} \end{matrix}\right\}$ the order of magnitude of either member. Here the words "higher," "lower," "increase," "diminution," are, of course, used in their algebraic sense.

(2°) Both members may be integrated between given limits as to any same variable, without diminution of index, unless the real part or the purely imag-

inary part of a member of the *R*-equality change sign during the integration, or unless a limit of integration for the *A*-equality be infinite. So, both members may be differentiated as to a variable without diminution of index except perhaps for particular values of the variable.

(3°) The suffix may be changed by raising it to any power if the index be divided by the exponent of that power: e. g. the copulas $\overset{n}{=}_k$ and $\overset{\frac{1}{2}n}{=}_{k^2}$ and $\overset{2n}{=}_{\sqrt{k}}$ are equivalent.

For example, we may evaluate as follows the elliptic function $\mathbf{p}u$ for $u \doteq 0$. By definition,

$$u \equiv \tfrac{1}{2}\int_{\mathbf{p}u}^{\infty} \frac{dy}{\sqrt{(y^3 - \frac{1}{4}g_2 y - \frac{1}{4}g_3)}}$$

$$\overset{2}{=}_{1/\mathbf{p}} \tfrac{1}{2}\int_{\mathbf{p}u}^{\infty} \frac{dy}{\sqrt{y^3}} = \mathbf{p}^{-\frac{1}{2}}u$$

$$\therefore\ \mathbf{p}u \overset{2}{=}_{1/\mathbf{p}} u^{-2},\ \overset{4}{=}_{u} u^{-2},\ \overset{42}{=} u^{-2},$$

$$= u^{-2} + \text{terms in } u^2 \text{ and higher powers.}$$

3. *Double Statements*, wherein all the upper or all the lower alternatives must be read together, are of course in common use; and they might often well replace the double column arrangement for exhibiting geometric or other duality. Sometimes there are three or more alternatives.

But occasionally there is for instance a double duality, wherein, to show what cases belong together, accents or other indices would best be used. I do not as yet see, however, that here there are two distinct uses for these marks, assignable respectively to upper and lower positions, as in Arts. 1 and 2.

A good instance of this would be Delambre's fourfold formula for the spherical triangle, when written thus:

$$\begin{matrix}\sin \\ \cos\end{matrix}\left\{\frac{A \overset{'}{\pm} B}{2}\right/\begin{matrix}\cos \\ \sin\end{matrix}\left\{\frac{C}{2} = \begin{matrix}\cos \\ \sin\end{matrix}\left\{\frac{a \overset{'}{\pm} b}{2}\right/\begin{matrix}\cos \\ \sin\end{matrix}\left\{\frac{c}{2}\right.\right.\right.\right.;$$

wherein we must use all the upper alternatives marked $'$, or else all the lower alternatives marked $'$, and so with the alternatives marked $''$.

When the choice turns upon the evenness or oddness of some number n, it is well to associate always the upper alternative with even values, and the lower with odd; writing the number as a suffix if necessary. Thus the general terms of the series $a_0 - a_1 + a_2 - a_3 + \ldots$ and $b_1 - b_2 + b_3 - \ldots$ would be $\pm_n a_n$ and $\pm_{n-1} b_n$, $= \mp_n b_n$, — or simply $\pm a_n$ and $\mp b_n$. With this convention the more cumbrous $(-1)^n$ is seldom needed: thus $a_0 + (-1)^{n-1} b_n$, and $(-1)^m a_m + (-1)^{n-1} b_n$, are simply $a_0 \mp b_n$ and $\pm_m a_m \mp_n b_n$. The laws of combination

[$\pm_m \pm_n$ is $\pm_{m+n}$, $\pm_m \mp_n$ is $\mp_{m+n}$, etc.] are obvious. A like convention would apply, if needed, to any k alternatives dependent upon the form of some number as to k; and also to symbols which, like i^k, are essentially ambiguous.

Analogous to double statements are statements involving elements joined by the conjunction (,) or (;). Here, too, indices may be sometimes useful; or we may, instead, make like conjunctions, in like type, to correspond. E. g., to express that the substitution of $(lx + my,\ l'x + m'y)$ for (x, y) transforms $(a, \beta, \gamma)(x, y)^2$, $\equiv ax^2 + 2\beta xy + \gamma y^2$, into $((a, \beta, \gamma)(l, l')^2;\ (a, \beta, \gamma)(l, l')(m, m');\ (a, \beta, \gamma)(m, m')^2)(x, y)^2$, wherein $(a, \beta, \gamma)(l, l')(m, m') \equiv a.lm + \beta(lm' + l'm) + \gamma.l'm'$, one might write

$$(a;\ \beta;\ \gamma)((l, m;\ l', m')(x, y))^2 = ((a, \beta, \gamma)(l, l';\ m, m')^2)(x, y)^2.$$

Restrictions or freedom in the order of correspondence of elements thus joined can be indicated much as with the double statements first considered.

E. g. $$(x, y, z) = (a, \beta, \gamma) = (a, b, c)' = 1, 2, 3$$

would seem to denote naturally that $x = a$, $y = \beta$, and $z = \gamma$; and that these are equal respectively to a, b, and c in some order elsewhere specified or determined, and to 1, 2, and 3 in some order that is not specified.

Sometimes it is not clear whether an ordinary double statement stands for two simultaneous statements or only for two alternative ones, and a like doubt may arise when commas or radicals, etc., are used. How best to remedy this would, perhaps, be worth considering.

The general principles of Arts. 1–3 would of course apply, at need, to the modifying of almost any simple or complex sign of quantity, operation, or relation.

4. *Order of members in a continued statement.* (cf. O. W. J., p. 8.)—In the absence of punctuation-marks, each copula in a continued statement naturally connects the two members just adjacent, and of course no others. This is too often forgotten, even by good writers; but, obviously, it is important in stating any chain of equalities or other assertions. A good example is in the above proof of Pascal's theorem.

Sometimes, however, it is convenient, after one or more transformations of a member, to go back, begin a new series of transformations of the first member, and equate the two final results thus obtained. This is naturally indicated by a comma before the first copula that connects back; e. g. $P =_1 Q \doteq_x R$, $=_{23} S$ should mean that $P = Q$ by reason of equation (1), and $P = S$ by (2) and (3), and $Q - R$ approaches the limit 0, or (preferably) Q/R approaches unity, as x approaches its limit; whence $R \doteq S$; while without the comma, P would $\doteq S$, and $R = S$.

So far all seems clear; but some nice questions remain, to which attention is invited. For instance, when a sign of inference precedes an unpunctuated continued statement, does it necessarily affect the first copula of that statement, and only the first copula? E. g., knowing that $\beta < \gamma$, and also, independently, that $\alpha < \beta + 3$, can we write $\beta < \gamma$, $\therefore \alpha < \beta + 3 < \gamma + 3$; or must it be $\beta < \gamma$; $\therefore \alpha, < \beta + 3, < \gamma + 3$, wherein two commas have the effect of a parenthesis?

5. *Other symbols.*—To the above, which relates to the natural modifications and uses of existing symbols, I add, with some diffidence, a brief mention of two or three further devices in general elementary notation; i. e. not in the notation of particular subjects such as derivatives or determinants. Some of them would admit of more development and theory than can be given here.

(*a.*) We need a logical copula to assert the dependence of one statement upon another without our first asserting the truth of either. This, $\therefore$ and $\because$ cannot do; but it is done by Prof. C. S. Peirce's $-\!\prec$ and $\succ\!-$; thus $P-\!\prec Q$, or $Q\succ\!- P$, means "If P be true, so is Q;" and $(A = B)-\!\prec(C = D)$, or $(C = D)\succ\!-(A = B)$, means "If $A = B$ then $C = D$." Such a statement neither implies nor denies its own converse; but we might write $P\succ\!-\!\prec Q$, for "If P be true, so is Q, and, conversely, if Q, then P;" i. e. "Statements P and Q are equivalent;" and this, or some other sign of equivalence, would often be wanted. Peirce's sign of equivalence is not $\succ\!-\!\prec$, but $=$; which is the best in his studies of Pure Logic, but which, of course, is preoccupied in Mathematics. Hence, it may be as well not to employ the forms $-\!\prec$, $\succ\!-$, if, as I think is the case, another form (the square half bracket) is already somewhat used for a quite analogous purpose, and is rather more convenient. When we write

$$C = D, \qquad [(8)$$

the marginal reference, [(8), is commonly a reason for the statement $C = D$; and if the formula (8), referred to, be $A = B$, we have thus virtually written $C = D\ [A = B$, in the sense $(C = D)\succ\!-(A = B)$ above; that is, we have the copulas [" *Is implied by* or necessary to;"]"*Implies* or is sufficient to;"][" *Is equivalent*, or both necessary and sufficient, to." These copulas [,],][, or their equivalents $\succ\!-$, $-\!\prec$, $\succ\!-\!\prec$, can, at need, be variously modified by indices. Thus we could form copulas for "Does not imply," "Implies but is not implied by," etc., analogous to the familiar "Is not equal to." When a logical copula connects only equations not continued and whose second members are zero, we might, for shortness, write only their first members; as, $A + B\ ]C, D$ for $A + B = 0]\ C = D = 0$; i. e. for $A + B = 0]\ C = 0, D = 0$.

(*b.*) As it is often convenient to refer to one or more specified members of an equation or other statement, or of a group of such, I denote them by

subscripts, as on p. 109 of Vol. III of this journal; e. g. (3_4) would be the fourth member of the continued statement, or group of statements (3); $(3_{1.4})$ would be equivalent to "(3_1) and (3_4);" and (3_{14}) or (3_{135}) would be the equality $(3_1) = (3_4)$ or $(3_1) = (3_3) = (3_5)$.

Expressions like (3_1) etc., or statements like (3_{14}) etc., may of course be used not merely as references, but as elements of new statements or expressions; e. g. of such as are suggested in Art. 5 (*a*).

(*c*) A few other conventions would seem desirable, so as for instance to fix more clearly the order in which the fundamental signs of operation shall combine (cf. O. W. J., I § 12), and to aid in indicating the motives and connection of the different steps in a discussion: but here, at present, I have no definite suggestions to make.

SOLUTIONS OF EXERCISES.

174

FIND the locus of the intersection of the altitudes of a triangle whose base and area are given.

SOLUTION.

Let the extremities of the base be $(\pm c, 0)$ and the vertex (x, h).

Then the declivity of the side is

$$m = \frac{h}{x - c},$$

and the declivity of the perpendicular to it

$$m' = \frac{y}{x + c}.$$

The condition of perpendicularity gives

$$hy = c^2 - x^2,$$

which represents a parabola with vertical axis passing through the extremities of the base. [*H. B. Newson; T. U. Taylor.*]

180

FIND the envelope of a system of circles each of which is seen from two fixed points under a constant angle.

SOLUTION.

Let the line joining the fixed points be the x-axis and its middle point be the origin. Let $2c$ be the distance between the points; let $\sin^{-1}(m)$ be half the constant angle.

It is evident that the locus of the centres of the system is the y-axis. Let (o, a) be the co-ordinates of the centre of any circle of the system. Then

$$r^2 = (a^2 + c^2)\, m^2,$$

and the equation to the circle is

$$x^2 + (y - a)^2 = (a^2 + c^2)\, m^2.$$

Differentiating with respect to the parameter a, we obtain

$$-2(y - a) = 2am^2;$$

whence $$a = \frac{y}{1 - m^2}.$$

This gives for the equation to the envelope

$$\frac{x^2}{m^2} - \frac{y^2}{1 - m^2} = c^2,$$

which represents an hyperbola with the fixed points as foci and $e = m^{-1}$.

[*H. B. Newson.*]

228

SUPPOSE a vessel in the form of an inverted frustum of a cone to be filled with water, and then tipped until as much water runs out as is possible without uncovering any part of the bottom; required, the volume of the water which remains.

SOLUTION.

Let R, r be the radii of the bases of the frustum, H altitude, a, b semi axis of the elliptic water surface. The supplemental cone has a volume

$$v = \tfrac{1}{3}\pi H \frac{r^3}{R - r}.$$

This added to the required volume gives an oblique cone whose altitude is

$$k = \frac{2hRr}{a(R - r)},$$

and whose volume, therefore, is

$$\tfrac{1}{3}\pi abk = \tfrac{1}{3}\pi H. \frac{b^3}{R - r}.$$

Hence the required volume is

$$V = \tfrac{1}{3}\pi H. \frac{b^3 - r^3}{R - r},$$

in which $$b = \sqrt{Rr},$$

as may be easily proved. [*B. E. Sheppard.*]

Also solved by *Mr. O. L. Mathiot.*

230

SOLVE the equations

$$x^2(y - z) = a,$$
$$y^2(z - x) = b,$$
$$z^2(x - y) = c.$$ [*Frank Morley.*]

SOLUTION.

$$a+b+c=-(y-z)(z-x)(x-y),$$

$$abc=x^2y^2z^2(y-z)(z-x)(x-y),$$

whence $$xyz=k$$

is known.

Accordingly $$ayz=k(xy-zx),$$

$$bzx=k(yz-xy);$$

whence $$ayz+bzx=k(yz-zx),$$

or $$\frac{x}{k-a}=\frac{y}{k+b},$$

$$\frac{y}{k-b}=\frac{z}{k+c},$$

$$\frac{z}{k-c}=\frac{x}{k+a},$$

are obtained in the same way, By means of these relations we eliminate y and z (say) from the first of the given equations and find x; thus

$$x^3\left[\frac{k+b}{k-a}-\frac{k-c}{k+a}\right]=a.$$

The value of k is $$k=\sqrt{\frac{-abc}{a+b+c}}.$$

[*T. U. Taylor.*]

231

IF we take products of n consecutive terms of the arithmetical series a, $a-d$, etc., commencing for the first product with the first term, for the second product with the second term, and so on; and then multiply these products by the coefficients in the expansion of $(1-x)^n$, the aggregate will be $n!d^n$, which is independent of the first term. [*W. W. Johnson.*]

SOLUTION.

If the n differences of a series of $n+1$ quantities be multiplied by the coefficients in the expansion of $(1-x)^{n-1}$, the aggregate is readily seen to be the same as that of the $n+1$ quantities multiplied by the coefficients of $(1-x)^n$.

Now the first product named in the question is

$$a(a-d)\ldots[a-(n-1)d],$$

and the second is

$$(a-d)\ldots[a-(n-1)d](a-nd).$$

Thus their difference is

$$nd\times(a-d)\ldots[a-(n-1)d].$$

Considering in like manner the differences between each pair of consecutive products, the theorem above shows that the aggregate in question is the product of nd into a similar aggregate in which n is changed to $n-1$, and d is unchanged. Thus denoting the aggregate by S_n, we have

$$S_n = nd \,.\, S_{n-1} = nd\,(n-1)\,d \,.\, S_{n-2} = \ldots = n!\,d^n,$$

since $S_1 = d$. [*W. W. Johnson.*]

Also solved by *Dr. R. A. Harris.*

234

In a regular heptagon $ABCDEFG$, show that

$$\frac{1}{AB} = \frac{1}{AC} + \frac{1}{AD}.$$ [*Frank Morley.*]

SOLUTION.

Let $x = \frac{1}{7}\pi$; then

$$\frac{1}{\sin x} - \frac{1}{\sin 3x} = \frac{2\cos 2x \sin x}{\sin x \sin 3x}$$

$$= \frac{2\cos 2x}{\sin 4x}$$

$$= \frac{1}{\sin 2x},$$

which was to be proved. [*T. U. Taylor.*]

Also solved by *Dr. R. A. Harris.*

235

Find the areas of the greatest and of the least rhombus inscribed in an ellipse. [*R. H. Graves.*]

SOLUTION I.

The central radius vector of an ellipse is given by the formula

$$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}$$

$$= \frac{1}{b^2}\,(1 - e^2\cos^2\theta).$$

If r_1, r_2 be orthogonal radii, the area of the rhombus of which they are semi-diagonals is

$$2r_1r_2 = \frac{2b^2}{\sqrt{[(1 - e^2\cos^2\theta)\,(1 - e^2\sin^2\theta)]}}.$$

This has its least value when

$$(1 - e^2\cos^2\theta)\,(1 - e^2\sin^2\theta) = z$$

is greatest, and its greatest value when this is least. But

$$s = 1 - e^2 + \tfrac{1}{4}e^4 \sin^2 2\theta$$

is greatest when $\sin 2\theta = 1$, and least when $\sin 2\theta = 0$. The corresponding values of the area are

$$Q' = \frac{2b^2}{1 - \frac{1}{2}e^2}, \qquad Q'' = \frac{2b^2}{\sqrt{1 - e^2}};$$

or

$$Q' = \frac{4a^2b^2}{a^2 + b^2}, \qquad Q'' = 2ab. \qquad [W.\ M.\ Thornton.]$$

SOLUTION II.

Let the ellipse be projected into a circle whose diameter is equal to the minor axis. Every parallelogram inscribed in the ellipse becomes a rectangle inscribed in the circle. In case of the rhombus, the sides being as 1 : 1, after projection the ratio of two adjacent sides of the rectangle may vary from 1 to $\frac{b}{a}$. Find now the greatest and least rectangles the ratios of whose adjacent sides lie within these limits.

The first is a square. For, draw axes through the centre of the rectangle parallel to its sides (the x-axis being parallel to the longer side), and let θ denote the angle which the diagonal makes with the x-axis. Then the area is

$$u = b^2 \sin\theta \cos\theta.$$

$$\therefore \frac{du}{d\theta} = b^2 \cos 2\theta;$$

and $\theta = 45^\circ$ causes $\frac{du}{d\theta}$ to vanish.

We can project this square into a rhombus only by projecting the diagonals into the major and minor axes. Hence, the area of the greatest rhombus is $2ab$.

u being an increasing function up to $\theta = 45^\circ$, the smaller we can take θ, the smaller will be the area. This value is $\theta = \tan^{-1}\left(\frac{b}{a}\right)$; i. e. the ratio of the two adjacent sides of our rectangle is $\frac{b}{a}$. Hence, the length of its longer side is $\frac{2ab}{\sqrt{a^2 + b^2}}$.

We can project this rectangle into a rhombus only by projecting the shorter sides into lines parallel to the major axis; the longer sides into lines parallel to the minor. This rhombus, then, is a square whose area is $\frac{4a^2b^2}{a^2 + b^2}$.

[*R. A. Harris.*]

EXERCISES.

251

LET $\Sigma_i p$ be the sum of any n ($<\frac{1}{2}m$) of the quantities $p_1, p_2, \ldots p_m$, taken at random; let $\Sigma_j p$ be a similar sum of n quantities not in $\Sigma_i p$; then will

$$\Pi\left(\frac{a_i + a_j}{2}\right)^{(2n)!(m-2n)!} > \Pi a_i^{n!(m-n)!},$$

where Π denotes the product taken with regard to all the possible combinations of the quantities $p_1, \ldots p_m$ involved in a_i and $a_i + a_j$ respectively.

[*Ormond Stone.*]

252

$f(x)$ and $\varphi(e^x)$ being rational algebraic functions of x and e^x respectively, show that

$$\int f(x)\,\varphi\,(e^x)\,dx$$

depends upon integrals of the three forms,

$$\int \frac{du}{\log u}, \quad \int \theta^m \tan\theta d\theta, \quad \int \frac{du}{(u+c)^r \log u}.$$

[*R. A. Harris.*]

253

ANY curve not of degree $3m$, which is symmetrical with regard to an equilateral triangle, is circular. [*Frank Morley.*]

254

USING the complex variable, the points z_1, z_2, z_3 will form an equilateral triangle if

$$z_1 + \omega z_2 + \omega^2 z_3 = 0$$

where ω is an imaginary cube root of 1. Hence, show that

(*a*) If O is a point inside an equilateral triangle ABC at which the sides subtend angles α, β, γ, then the triangle whose sides are equal to OA, OB, OC will have angles equal to $\alpha - \frac{1}{3}\pi$, $\beta - \frac{1}{3}\pi$, $\gamma - \frac{1}{3}\pi$.

(*b*) If equilateral triangles are described on the sides of a plane polygon, all in the same sense, the centre of mean position of the corners of the triangles which do not coincide with those of the polygon, will coincide with that of the corners of the polygon. [*Frank Morley.*]

255

In a regular polygon of m sides let r_1, r_2, etc., be the lengths of the lines from one corner to the others taken in order; then

$$1 + \frac{r_{3r-m}}{r_r} = \frac{r_{2m-4r}}{r_{m-2r}},$$

where $r > \frac{1}{3}m$, $< \frac{1}{2}m$. [*Frank Morley.*]

256

Prove the construction for the tangent at a cusp A of a quintic having cusps at A, B, C, D, E. Let BC, DE cut at P; let AQ be the fourth harmonic of AB, AC, AP in the involution defined by AB, AC and AD, AE; let AR be the ray conjugate to AQ; let AS be the fourth harmonic of AP, AQ, AR; let AT be the ray of the involution conjugate to AS; then AT is the required tangent. [*Frank Morley.*]

257

Given three harmonic motions of equal periods, amplitudes a, b, c, and at right angles. Find the surface which includes every phase of the resulting curvilinear motion when the ratio of the rates of change of phase is given. [*W. S. Franklin.*]

258

Given a limited looped curve consisting of connected portions of known curves. Find two series of harmonic motions at right angles which, when compounded with a uniform motion of translation, will be equivalent to the motion of a point traveling along the curve with uniform velocity. [*W. S. Franklin.*]

259

Find the general equations to the curve assumed by the water jet of a "Barker's Mill," the axis of rotation not being vertical. The velocity of issue, however, being assumed constant. [*W. S. Franklin.*]

260

A's coefficient of veracity is p, that of B is equally likely to be v or $1 - v$. Required: (1) the probability of the truth of a statement to which they agree; (2) the probability of the truth of a statement which A makes and B denies. [*Henry Heaton.*]

261

Find the locus described by a point on the connecting rod of a steam engine when the piston rod produced does not cut the crank circle. [*T. U. Taylor.*]

262

Determine the volume produced by revolving the curve $y^{\frac{1}{3}} - x^{\frac{1}{3}} = a^{\frac{1}{3}}$ about $x - a = 0$. [*O. Root, Jr.*]

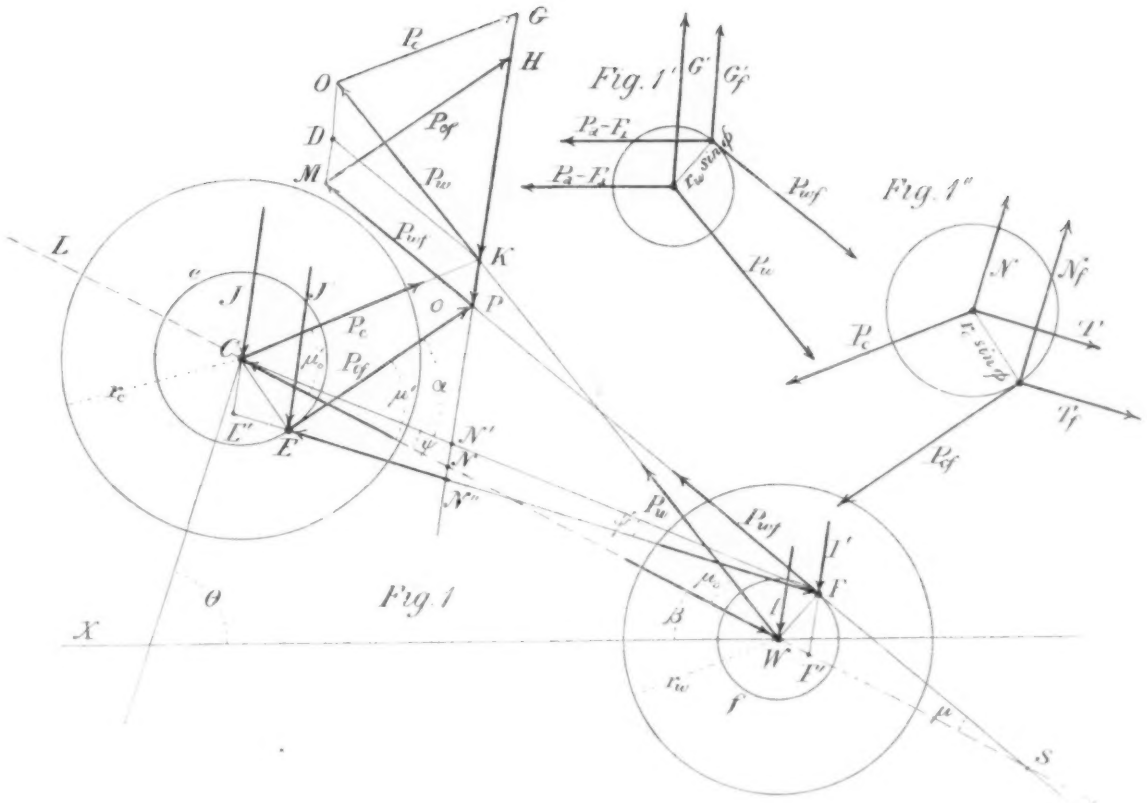
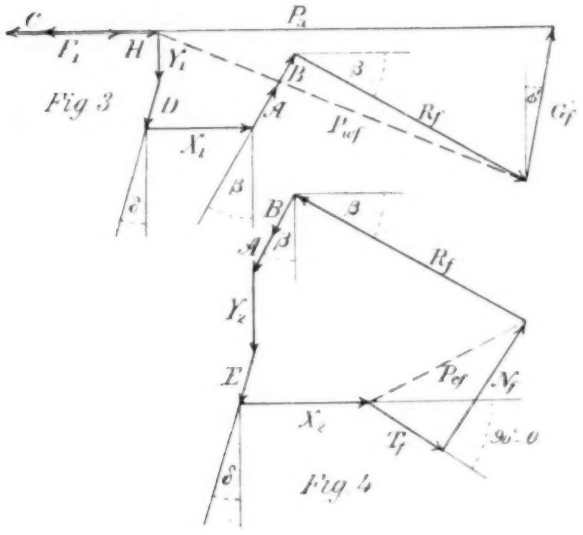


Fig. 1
Fig. 1'
Fig. 1''
Fig 2
Fig. 2'


C
P_a
P_1
H
Y_1
B
β
Fig 3
D
A
B_f
G_f
P_{af}
X_1
β
ε
B
β
A
β
B_f
Y_2
E
P_{ef}
N_f
X_2
T_f
$90^\circ - \theta$
ε
Fig 4